Deutsch-Jozsa Algorithm

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1. Deutsch-Jozsa Problem

Boolean function: $f : \{0,1\}^n \longrightarrow \{0,1\}$

<u>Constant</u> $\forall x \in \{0,1\}^n \; f(x) = 0$ or $\forall x \in \{0,1\}^n \; f(x) = 1$

<u>Balanced</u> $\sum_{x \in \{0,1\}^n} f(x) = 2^{n-1}$

Input: function f which is either Constant or Balanced

Output: $\begin{cases} 0 & \text{if } f \text{ is Constant} \\ 1 & \text{if } f \text{ is Balanced} \end{cases}$

<u>Classical Algorithm</u>

```
def dj(f, n):
    out = f(0)
    for x in range(1, 2^(n-1) + 1):
        if f(x) != out:
            return 1
    return 0
```

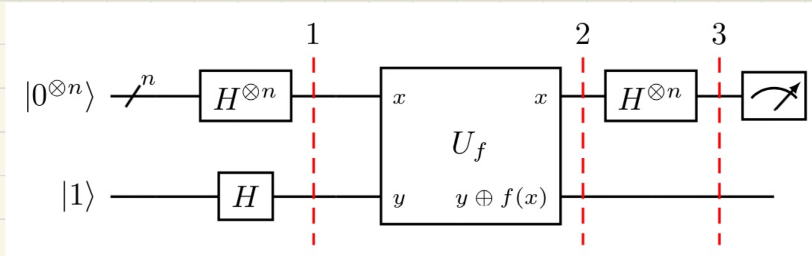
$O(2^{n-1})$

2. Quantum Algorithm

Oracle

x Input register

y Output register



$\oplus$ sum modulus 2

Step 1. Prepare 2 quantum registers

$$|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$$

Step 2 Apply Hadamard gates

$$|\psi_1\rangle = (H|0\rangle)^{\otimes n} (H|1\rangle)$$

$$= \frac{1}{\sqrt{2^{n+1}}} (|0\rangle + |1\rangle)^{\otimes n} (|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{x=0}^{2^n-1} |x\rangle (|0\rangle - |1\rangle)$$

Step 3 Apply the quantum oracle

$$U_f : \{0,1\}^{n+1} \longrightarrow \{0,1\}^{n+1}$$

$$|x\rangle|y\rangle \longmapsto |x\rangle|y \oplus f(x)\rangle$$

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

$$|\psi_2\rangle = U_f|\psi_1\rangle = U_f\left[\frac{1}{\sqrt{2^{n+1}}}\sum_{x=0}^{2^n-1}|x\rangle\left(|0\rangle - |1\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2^{n+1}}}\sum_{x=0}^{2^n-1}|x\rangle\left(|f(x)\rangle - |1 \oplus f(x)\rangle\right)$$

$$= \frac{1}{\sqrt{2^{n+1}}}\sum_{x=0}^{2^n-1}|x\rangle(-1)^{f(x)}\left(|0\rangle - |1\rangle\right)$$

Step 4. Apply a Hadamard to x register

$$|x\rangle = |x_n \, x_{n-1} \cdots x_1\rangle$$

$$H^{\otimes n}|011\rangle = |+--\rangle = (|0\rangle+|1\rangle)(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)$$

$$= |000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle$$

Parity sum = Parity$(\Sigma x_i y_i)$

$$x \cdot y = \overbrace{\underbrace{x_1 y_1}_{\text{Apply } x \text{ Mask to } y} \oplus x_2 y_2 \oplus \cdots \oplus x_n y_n}$$

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} (-1)^{x \cdot y} |y\rangle$$

$$|\psi_3\rangle = H^{\otimes n}|\psi_2\rangle_x = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} H^{\otimes n}|x\rangle (-1)^{f(x)}$$

$$= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{x \cdot y} |y\rangle (-1)^{f(x)}$$

$$= \frac{1}{2^n} \sum_{y=0}^{2^n-1} \left[\sum_{x=0}^{2^n-1} (-1)^{f(x)} (-1)^{x \cdot y} \right] |y\rangle$$

Step 5. Measure the first register

$$|\psi_3\rangle = \alpha_{|00\ldots0\rangle}|00\ldots0\rangle + \alpha_{|00\ldots1\rangle}|00\ldots1\rangle + \ldots + \alpha_{|11\ldots1\rangle}|11\ldots1\rangle$$

$$P\left(|0\rangle^{\otimes n} \,\big|\, |\psi_3\rangle\right) = \left|\alpha_{|0\rangle^{\otimes n}}\right|^2 = \left|\frac{1}{2^n}\sum_{x=0}^{2^n-1}(-1)^{f(x)}\right|^2$$

$$= \begin{cases} 1 & f \text{ is constant} \\ 0 & f \text{ is balanced} \end{cases}$$